

NON-FORMALITY OF PLANAR CONFIGURATION SPACES IN CHARACTERISTIC TWO

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ABSTRACT. Let $F_k(\mathbb{R}^2)$ be the ordered configuration space of k distinct points in the plane. We prove that its singular cochain algebra with coefficients in \mathbb{Z}_2 is not formal as a differential graded algebra for $k \geq 4$.

1. INTRODUCTION

The notion of formality of a topological space X is usually introduced in the sense of rational homotopy theory via some commutative differential algebra model in characteristic 0 of X , like the Sullivan-deRham algebra $A_{PL}(X)$, or the algebra of differential forms if X is a manifold. However one can define formality over any coefficient ring R in the non-commutative context, by requiring the algebra of R -valued cochains $C^*(X, R)$ to be quasi-isomorphic to the cohomology $H^*(X, R)$. This means that $C^*(X, R)$ is connected to its cohomology $H^*(X, R)$ by a zig-zag of homomorphisms of differential graded associative R -algebras inducing isomorphisms in cohomology. If R is a field then this property depends only on the characteristic of R . If R is a field of characteristic 0 then the formality in the associative sense is equivalent to the usual commutative formality by a recent result of Saleh [15].

Let us consider the euclidean configuration spaces

$$F_k(\mathbb{R}^n) = \{(x_1, \dots, x_k) \in (\mathbb{R}^n)^k \mid x_i \neq x_j \text{ for } i \neq j\}.$$

Kontsevich and Lambrechts-Volic have proved their formality in characteristic zero.

Theorem 1.1. [10, 11] *The configuration space $F_k(\mathbb{R}^n)$ is formal over \mathbb{R} for any k, n .*

The case of the configuration spaces in the plane $F_k(\mathbb{R}^2)$ had been proved much earlier by Arnold [1].

What can be said about formality over the integers, or in positive characteristic?

It is not difficult to prove the following result.

Theorem 1.2. *The configuration space $F_k(\mathbb{R}^n)$ is intrinsically formal over \mathbb{Z} if $n \geq k$.*

This means that any space with the same cohomology ring as the configuration space is formal over \mathbb{Z} . The case $n < k$ is open in general. A special case is $F_3(\mathbb{R}^2) \simeq S^1 \times (S^1 \vee S^1)$ that is formal over \mathbb{Z} .

Geoffroy Horel has announced a proof of \mathbb{Z}_p -formality of $F_k(\mathbb{R}^n)$ for any $n > 2$ even, using étale cohomology.

One might think that formality over \mathbb{Z} holds for all configuration spaces as in the rational case.

However we found the following surprising result:

Theorem 1.3. *The configuration space $F_k(\mathbb{R}^2)$ is not formal over \mathbb{Z}_2 for any $k \geq 4$.*

This immediately implies non-formality over the integers.

Corollary 1.4. *$F_k(\mathbb{R}^2)$ is not formal over \mathbb{Z} for any $k \geq 4$.*

We approach this question by obstruction theory following the work by Halperin-Stasheff [9] for cdga (commutative differential graded algebras) in characteristic 0 and its extension to the non-commutative case by El Haouari [7]. We implemented the computation of the non-trivial obstruction in MATLAB. The relevant library is attached.

In a follow up paper we will deal with configuration spaces in \mathbb{R}^3 .

The paper is organized as follows: in section 2 we recall the presentation of the cohomology rings of the configuration spaces and of their Koszul dual, the Yang Baxter algebras. In section 3 we define formality of an algebra and recall the obstruction theory to it by Halperin-Stasheff via bigraded models. In section 4 we describe the Barratt-Eccles-Smith simplicial model for euclidean configuration spaces. In section 5 we apply the obstruction theory and prove that $F_k(\mathbb{R}^2)$ is not formal over \mathbb{Z}_2 for $k \geq 4$. In section 6 we interpret the obstruction class in Hochschild cohomology. Finally in the appendix we describe the attached software in MATLAB.

2. COHOMOLOGY OF CONFIGURATION SPACES AND YANG-BAXTER ALGEBRAS

We recall the presentation of the cohomology ring of the configuration spaces $F_k(\mathbb{R}^n)$. Then we describe its Koszul dual, the Yang-Baxter algebra, and its geometric interpretation.

Consider the direction map $\pi_{i,j} : F_k(\mathbb{R}^n) \rightarrow S^{n-1}$ from the i -th to the j -th particle given by $\pi_{i,j}(x_1, \dots, x_k) = (x_j - x_i)/|x_j - x_i|$.

Let $\iota \in H^{n-1}(S^{n-1})$ be the standard fundamental class. We write

$$A_{ij} = \pi_{i,j}^*(\iota) \in H^{n-1}(F_k(\mathbb{R}^n)).$$

The following computation of the cohomology ring is due to Arnold in the case $n = 2$ and Fred Cohen for $n > 2$.

Theorem 2.1. (Arnold, Cohen) [1, 5] *The cohomology ring $H^*(F_k(\mathbb{R}^n))$ for $n \geq 2$ has a presentation with $(n-1)$ -dimensional generators A_{ij} for $1 \leq i \neq j \leq n$ and relations*

- (1) $A_{ij} = (-1)^n A_{ji}$
- (2) $A_{ij}^2 = 0$
- (3) (Arnold) $A_{ij}A_{jk} + A_{jk}A_{ki} + A_{ki}A_{ij} = 0$ for $i \neq j \neq k \neq i$

Thus up to the grading this ring depends only on the parity of n .

Corollary 2.2. *The cohomology groups $H^*(F_k(\mathbb{R}^n))$ are torsion free, and have a graded basis provided by the set*

$$\{A_{i_1 j_1} \cdots A_{i_t j_t} \mid j_1 < \cdots < j_t, i_t < j_t \quad \forall t\}$$

Corollary 2.3. *The Poincare series of the configuration space is*

$$P(F_k(\mathbb{R}^n)) = \prod_{m=1}^{k-1} (1 + mx^{n-1})$$

Notice that $F_k(\mathbb{R}^n)$ is the total space of a tower of fibrations with fibers homotopic to $\vee_m S^{n-1}$ for $m = 1, \dots, k-1$. Thus the additive structure does not see the twisting of the fibrations, but the cohomology ring does.

A remarkable fact is that the cohomology of the configuration spaces is a Koszul algebra (see example 32 in [3]). The following holds with coefficients in \mathbb{Z} .

Proposition 2.4. *The Koszul dual algebra of $H^*(F_k(\mathbb{R}^n))$ for $n \geq 2$ is the Yang-Baxter algebra $YB_k^{(n)}$ generated by classes B_{ij} of degree $n-2$, for $1 \leq i \neq j \leq k$ under the relations*

- (1) $B_{ij} = (-1)^n B_{ji}$
- (2) (Yang-Baxter)

$$[B_{ij}, B_{jk}] = [B_{jk}, B_{ki}] = [B_{ki}, B_{ij}] \text{ for } i \neq j \neq k \neq i$$

- (3) $[B_{ij}, B_{rt}] = 0$ when $\{i, j\} \cap \{r, t\} = \emptyset$

Clearly this algebra up to the grading depends only on the parity of n .

Proposition 2.5. [6] *The Yang-Baxter algebra is torsion free and has a graded basis*

$$\{B_{i_1 j_1} \cdots B_{i_l j_l} \mid j_1 \leq \cdots \leq j_l, i_t < j_t \forall t\}$$

The Yang-Baxter algebra has the following geometric meaning for $n > 2$: it is the Pontrjagin ring of the loop space of the configuration space $F_k(\mathbb{R}^n)$. Notice that under this hypothesis $F_k(\mathbb{R}^n)$ is $(n-2)$ -connected.

Let $A_{ij}^* \in H_{n-1}(F_k(\mathbb{R}^n))$ be the homology basis dual to A_{ij} . By the Hurewicz theorem and adjointness we have isomorphisms

$$H_{n-2}(\Omega F_k(\mathbb{R}^n)) \cong \pi_{n-2}(\Omega F_k(\mathbb{R}^n)) \cong \pi_{n-1}(F_k(\mathbb{R}^n)) \cong H_{n-1}(F_k(\mathbb{R}^n))$$

Let

$$B'_{ij} \in H_{n-2}(\Omega F_k(\mathbb{R}^n))$$

be the class corresponding to A_{ij}^* under the composite isomorphism.

Theorem 2.6. (Cohen-Gitler, Fadell-Hussein) [6, 8] *There is an isomorphism of algebras*

$$H_*(\Omega F_k(\mathbb{R}^n)) \cong YB_k^{(n)}$$

for $n > 2$ sending B'_{ij} to B_{ij} .

Corollary 2.7. *The Poincare series of $\Omega F_k(\mathbb{R}^n)$ for $n > 2$ is*

$$P(\Omega F_k(\mathbb{R}^n)) = \prod_{m=1}^{k-1} (1 - mx^{n-1})^{-1}$$

This follows also from the homotopy equivalence

$$\Omega F_k(\mathbb{R}^n) \simeq \prod_{m=1}^{k-1} \Omega(\vee_1^m S^{n-1})$$

that is not multiplicative in general.

For $n = 2$ the (ungraded) algebra $YB_k = YB_k^{(2)}$ has also a similar geometric meaning that we explain. Let us start from the following well known result.

Proposition 2.8. *The configuration space $F_k(\mathbb{R}^2)$ is a classifying space of the pure braid group on k strands $P\beta_k$.*

Now consider the descending central series of $P\beta_k$ defined by

$$G_1 = P\beta_k \text{ and } G_i = [G_1, G_{i-1}].$$

The direct sum of the subquotients forms a Lie algebra

$$\mathcal{L}_k = \oplus_i (G_i / G_{i+1})$$

under the bracket induced by taking commutators. Let U denote the universal enveloping algebra functor.

Theorem 2.9. [6] *There is an isomorphism $YB_k \cong U\mathcal{L}_k$ sending B_{ij} to the Artin generator \mathcal{A}_{ij} of the pure braid group.*

3. FORMALITY AND BIGRADED MODELS

We present the definition of formality in rational homotopy theory and in the non-commutative sense. We then proceed to describe the non-commutative versions of the bigraded models by Halperin and Stasheff, due to El Haouari.

Definition 3.1. [9] A topological space X is rationally *formal* if there is a zig-zag of quasi-isomorphisms of commutative differential graded algebras connecting the Sullivan-deRham algebra $A_{PL}(X)$ to its cohomology $H^*(A_{PL}(X)) \cong H^*(X, \mathbb{Q})$ equipped with the trivial differential.

If X is a manifold this is equivalent to the existence of a zig-zag of quasi-isomorphisms between the algebra of de-Rham forms $\Omega^*(X)$ and its real cohomology $H^*(X, \mathbb{R})$. If X is a complex manifold we might use the ring of complex differential forms and cohomology with complex coefficients. Namely the notion of formality in characteristic zero does not depend on the field for connected spaces of finite type (Theorem 6.8 in [9]). The same is true in positive characteristic, by a similar proof following the obstruction theory in [7].

Kontsevich [10] and Lambrechts-Volic [11] have proved that the configuration space $F_k(\mathbb{R}^n)$ is formal over \mathbb{R} for any k and n (Theorem 1.1).

This approach uses the commutative algebra of real PA forms and proves also the formality of the little discs operads.

Arnold had easily proved the formality of $F_k(\mathbb{C})$ [1]. His quasi-isomorphism embeds $H^*(F_k(\mathbb{C}), \mathbb{C})$ into the algebra of complex differential forms $\Omega_{\mathbb{C}}^*(F_k(\mathbb{C}))$ sending A_{ij} to $d(z_i - z_j)/(z_i - z_j)$.

We turn now to the non-commutative case.

Definition 3.2. Let R be a commutative ring. A topological space X is R -formal, or formal over R , if the algebra of singular cochains $C^*(X, R)$ is connected to its cohomology $H^*(X, R)$ by a zig-zag of quasi-isomorphisms of differential graded R -algebras.

For spaces of finite type with torsion free homology \mathbb{Z} -formality is universal as it implies R -formality for any ring R .

Here is an application of formality to computations: let ΩX and LX be respectively the based and the unbased loop space of X . Let HH_* denote the Hochschild homology functor from differential graded R -algebras to graded R -modules.

Proposition 3.3. *Let X be a simply connected R -formal space of finite type. Then there are isomorphisms of graded R -modules*

$$\begin{aligned} \mathrm{Tor}_{H^*(X,R)}(R, R) &\cong H^*(\Omega X, R) \\ HH_*(H^*(X, R)) &\cong H^*(LX, R) \end{aligned}$$

This follows from the collapse of the relevant Eilenberg-Moore spectral sequences.

The theory of R -formality is studied by El Haouari [7], who extends the obstruction theory by Halperin-Stasheff to the non-commutative case. The key point is the existence of a bigraded model that is then deformed to an actual model. A bigraded module $V = \oplus V_k^n$ has an upper dimensional grading, the *degree*, and a lower grading, the *resolution level*. The tensor algebra TV on a bigraded module inherits a bigrading by additivity. We work in the category of *cochain* differential graded algebras over a commutative ring R with upper grading, such that the differential is homogeneous of degree 1. A bigraded R -algebra has a differential that is also homogeneous with respect to the lower grading, and lowers it by 1. The tensor algebra on a bigraded module inherits a bigrading as well, and its cohomology too.

We say that a R -DGA A is connected if $H^0(A) = R$, generated by the unit of A .

Proposition 3.4. *(2.1.1. of [7]) Let R be a field. Given a connected graded R -algebra H , there exists a bigraded R -module V and a differential d on the tensor algebra $T(V)$ together with a quasi-isomorphism*

$$\rho : (T(V), d) \rightarrow (H, 0)$$

such that the homology in positive resolution level vanishes:

$$H_+(T(V), d) = 0.$$

Moreover $V_0 = H$ and $\rho|_{V_0}$ is the identity. The algebra $(T(V), d)$ is unique up to isomorphism and is called the bigraded model of H .

Remark 3.5. Proposition 3.4 works also over a commutative ring R when H^i and $\mathrm{Tor}_H^i(R, R)$ are free R -modules for each i . In particular it holds for $R = \mathbb{Z}$ and $H = H^*(F_k(\mathbb{R}^n))$.

Example 3.6. In the case of the configuration space cohomology $H = H^*(F_k(\mathbb{R}^n))$, the lower grading of the Yang-Baxter algebra $YB_k^{(n)} = H^1$ is the length of words in the standard generators. The suspended filtered dual $V = s(YB_k^{(n)})^*$ generates the bigraded model $(T(V), d) = B(YB_k^{(n)})^*$ that is the filtered dual of the bar construction. In other words the differential is the derivation induced by the coproduct $d : V \xrightarrow{\mu^*} V \otimes V \subset T(V)$ dual to the multiplication μ of $YB_k^{(n)}$. The resolution map $\Psi : (T(V), d) \rightarrow H$ is then chosen by

$$\Psi(B_{ij}^*) = A_{ij},$$

$$\Psi((B_{i_1 j_1} \dots B_{i_l j_l})^*) = 0 \quad \forall l > 1$$

We do not need to consider filtered duals for $n > 2$ since in that case $YB_k^{(n)}$ is of finite type, i.e. finite dimensional in each degree.

A bigraded module A has an induced filtration level by $F_k(A) = \oplus_{i \leq k} A_i^*$. We say that the differential lowers filtration level by h if $d(F_k(A)) \subset F_{k-h}(A)$.

The key idea of Halperin-Stasheff is to deform the bigraded model of the cohomology $H(A)$ of a DGA A in order to get a model for the algebra itself.

Theorem 3.7. (2.2.2 of [7]) *Let A be a connected DGA over a field R . Let $(T(V), d)$ be the bigraded model of $H^*(A)$. Then there is a differential D on $T(V)$ and a quasi-isomorphism*

$$\pi : (T(V), D) \longrightarrow A$$

such that $D - d$ lowers filtration level by 2. If $\pi' : (T(V), D') \rightarrow A$ is another solution to this problem, then there exists an isomorphism

$$\phi : (T(V), D) \longrightarrow (T(V), D')$$

such that $\phi - id$ lowers filtration level by 1, and $\pi' \circ \phi \simeq \pi$. Any solution is called a model of A .

The following result completes the theory, compare Theorem 5.3 in [9].

Proposition 3.8. *Let A, B be connected DGA's over a field R with an isomorphism $\bar{\phi} : H(A) \cong H(B)$. Let $(T(V), D_A), (T(V), D_B)$ be the respective models. Then A and B are quasi-isomorphic if and only if there exists an isomorphism $\phi : (T(V), D_A) \rightarrow (T(V), D_B)$ such that $\phi - id$ lowers filtration.*

We give a first immediate application.

Definition 3.9. A space X is intrinsically R -formal if it is R -formal, and any other space with the same cohomology R -algebra is also R -formal.

Theorem 3.10. *The configuration space $F_k(\mathbb{R}^n)$ is intrinsically \mathbb{Z} -formal if $n \geq k$.*

Proof. We can assume $n > 2$ since intrinsic formality of $S^1 \simeq F_2(\mathbb{R}^2)$ is easy. The generators of the bigraded model $(T(V), d)$ for $H = H^*(F_k(\mathbb{R}^n), R)$ are in degree

$$n-1, 2n-3, \dots, 1+i(n-2), \dots$$

since $V \cong s(YB_n)^*$, and respective level

$$0, 1, 2, 3, \dots, i-1, \dots$$

Consider the model $(T(V), D)$ for the singular cochain algebra $C^*(Y, R)$ of a space Y with $H^*(Y, R) \cong H$. We have that $D - d$ lowers the filtration by 2. All generators are in degree 1 mod $n-2$, and so their differentials are in degree 2 mod $n-2$. For a generator v of level $i-1$, and thus degree $1+i(n-2)$, a homogeneous non-zero monomial can be a term of $(D-d)(v)$ only if it has degree 2 mod $n-2$. But quadratic monomials $v_a v_b$, with $v_a \in V_{a-1}$ and $b \in V_{b-1}$, have to be ruled out because they would have degree $2+i(n-2) = [1+a(n-2)] + [1+b(n-2)]$, thus filtration level $(a-1) + (b-1) = i-2$, contradicting that $D-d$ lowers filtration by two. The monomial is then the product of at least n generators, that is in degree at least $n + n(n-2) = 2 + (n+1)(n-2)$, so $i > n$. Then $d = D$ on $V \leq n$. The top cohomology of the configuration space (and Y) is in dimension $(k-1)(n-1) < n(n-1)$, so there is no obstruction to extending the restriction $(T(V_{\leq n}), D) \rightarrow H$ inductively to a quasi-isomorphism $(T(V), D) \rightarrow H$. \square

A special case not covered by the theorem is that of $F_3(\mathbb{R}^2) \simeq S^1 \times (S^1 \vee S^1)$, which is \mathbb{Z} -formal, since wedges and products preserve formality [9].

In order to prove the non-formality of $F_k(\mathbb{R}^2)$ (Theorem 1.3) over \mathbb{Z}_2 for $k > 3$ we introduce simplicial versions of these configuration spaces in the next section.

4. THE BARRATT-ECCLES COMPLEX

We recall the Smith simplicial model for planar configuration spaces, obtained by filtering the Barratt-Eccles complex [2]. For simplicity we write $F_k := F_k(\mathbb{R}^2)$, and we adopt \mathbb{Z}_2 -coefficients throughout although we do not state it explicitly.

Definition 4.1. The full Barratt-Eccles complex on k elements is the geometric bar construction of the symmetric group on k letters $W\Sigma_k$.

This is a simplicial set with $(W\Sigma_k)_l = (\Sigma_k)^{l+1}$, so any element is a string of permutations. The face operators delete a single permutation in the string, and the degeneracies double a permutation in the string.

The symmetric group Σ_k acts diagonally levelwise.

In particular $W\Sigma_2$ has 2 non-degenerate simplices in each degree that are

$$(12|21|12|..), \quad (21|12|21|..).$$

Consider the sub-simplicial set $\mathcal{F}_t(W\Sigma_2) \subset W\Sigma_2$ spanned by non-degenerate simplices of dimension at most $t-1$.

The geometric realization $|\mathcal{F}_t(W\Sigma_2)|$ has the homotopy type of the $(t-1)$ -sphere.

We can extend the filtration to any $k > 2$ as follows.

There are simplicial versions of the projections seen in section 2, that we still denote

$$\pi_{ij} : W\Sigma_k \rightarrow W\Sigma_2, \text{ for } 1 \leq i \neq j \leq k.$$

Levelwise these maps are powers of the function $\pi_{ij} : \Sigma_k \rightarrow \Sigma_2$ defined by

$$\pi_{ij}(\sigma) = \begin{cases} (12) & \text{if } \sigma^{-1}(i) < \sigma^{-1}(j) \\ (21) & \text{if } \sigma^{-1}(i) > \sigma^{-1}(j). \end{cases}$$

Definition 4.2. The t -th stage $\mathcal{F}_t(W\Sigma_k)$ of the filtration of the Barratt-Eccles complex is defined by

$$\mathcal{F}_t(W\Sigma_k)_l = \bigcap_{i,j} \pi_{ij}^{-1}(\mathcal{F}_t(W\Sigma_2)_l)$$

Proposition 4.3. [2] *The geometric realization of the simplicial set $\mathcal{F}_t(W\Sigma_k)$ has the homotopy type of the configuration space $F_k(\mathbb{R}^t)$, and the realization of π_{ij} can be identified up to homotopy to the projection $\pi_{ij} : F_k(\mathbb{R}^t) \rightarrow S^{t-1}$ considered in section 2.*

Let N^* denote the normalized cochain functor from simplicial sets to differential graded algebras (over \mathbb{Z}_2 in our case).

Definition 4.4. The Barratt-Eccles DG-algebra with *complexity* t and *arity* k is

$$\mathcal{E}_t^*(k) := N^*(\mathcal{F}_t(W\Sigma_k)).$$

Here k is the *arity* of the Barratt-Eccles operad, that is equal to the number of points in the configuration. The upper index indicates the degree. We are interested mainly in $t = 2$.

Observe that each $\mathcal{E}_t^i(k)$ is a free module over $\mathbb{Z}_2[\Sigma_k]$, so that $k!$ divides its dimension over \mathbb{Z}_2 (this holds for any coefficient ring). For fixed t the top dimensional non-trivial vector space is in degree $(t-1)\binom{k}{2}$.

To get a flavour we present some cases of the generating polynomial

$$P_t^k(x) = \sum_i \dim_{\mathbb{Z}_2}(\mathcal{E}_t^i(k))x^i$$

$$P_2^2(x) = 2(1+x)$$

$$P_2^3(x) = 6(1+5x+6x^2+2x^3)$$

$$P_2^4(x) = 24(1+23x+104x^2+196x^3+184x^4+86x^5+16x^6)$$

$$P_3^2(x) = 2(1+x+x^2)$$

$$P_3^3(x) = 6(1+5x+25x^2+60x^3+70x^4+38x^5+8x^6)$$

$$P_3^4(x) = 24(1+23x+529x^2+5550x^3+30214x^4+97048x^5+\dots)$$

Let us denote the elements of the symmetric group Σ_3 by the following letters:

$$A = (123), B = (132), C = (213), D = (231), E = (312), F = (321).$$

Then $\mathcal{E}_2^2(3)$ is the free $\mathbb{Z}_2[\Sigma_3]$ -module on the 6 simplices

$$(A|D|F)^*, (A|E|F)^*, (A|C|F)^*, (A|C|D)^*, (A|B|F)^*, (A|B|E)^*,$$

hence it is 36-dimensional over \mathbb{Z}_2 .

By Proposition 4.3 we have quasi-isomorphic DGA's $\mathcal{E}_2^*(k) \simeq C^*(F_k(\mathbb{R}^2))$.

Proposition 4.5. *Non-formality of $\mathcal{E}_2^*(k)$ for $k \geq 4$ is equivalent to Theorem 1.3.*

Consider the 1-cochain

$$\omega_{ij} = \pi_{ij}^*((12|21)^*) \in \mathcal{E}_2^1(k).$$

For example for $k = 3$ ω_{ij} is the sum of 9 generators, and for $k = 4$ of 144 generators.

Consider the 1-cochain in $\mathcal{E}_2^1(3)$

$$Ar := (F|E)^* + (E|F)^* + (E|D)^* + (E|C)^* + (E|A)^* + (E|B)^* + (D|E)^* + (C|E)^* + (A|E)^*.$$

Lemma 4.6. *A resolution of the Arnold relation for 3 points on the cochain level is given by*

$$d(Ar) = \omega_{13}\omega_{12} + \omega_{23}\omega_{12} + \omega_{23}\omega_{13}$$

Proof. Let us compute the pairing $\langle d(Ar), X \rangle = \langle Ar, \partial(X) \rangle$ for basis generators $X \in \mathcal{E}_2^2(3)^*$. The cochain Ar is the sum of all generators of the form $(E|x)^*$ with $E \neq x \in \Sigma_3$, and of the form $(x|E)^*$ with $x \neq B$ and $x \neq E$. Thus $d(Ar)$ vanishes on all generators not containing E , and also on chains $(E|x|y)$, since in complexity 2 there are no chains of the form $(E|x|E)$, and

$$\langle Ar, \partial(E|x|y) \rangle = \langle Ar, (E|x) + (E|y) \rangle = 1 + 1 = 0 \pmod{2}$$

If E is in the middle position, $d(Ar)$ pairs non-trivially with chains of the form $(B|E|x)$. Since 1 and 3 swap going from B to E , only 2 can move with respect to 1 and 3, comparing $E = (312)$ and x , hence the possibilities are $x = D = (231)$ and

$x = F = (321)$. Finally $d(Ar)$ pairs non-trivially on chains of the form $(x|B|E)$, and again only 2 can move from x to B so that $x = A$ or $x = C$. This proves that

$$(4.1) \quad d(Ar) = (B|E|D)^* + (B|E|F)^* + (A|B|E)^* + (C|B|E)^*.$$

The cochain $\omega_{13}\omega_{12}$ pairs non trivially with chains $(x|y|z)$ with (13) and (12) substrings of x , (31) and (12) of y , and (31) and (21) of z . This forces $x = (123)$ or $x = (132)$. In the first case $y = (312)$ and $z = (321)$. In the second $y = (312)$, and either $z = (321)$ or $z = (231)$, hence

$$(4.2) \quad \omega_{13}\omega_{12} = (A|E|F)^* + (B|E|F)^* + (B|E|D)^*.$$

By similar considerations we obtain

$$(4.3) \quad \omega_{23}\omega_{12} = (A|B|F)^* + (A|E|F)^*,$$

$$(4.4) \quad \omega_{23}\omega_{13} = (A|B|E)^* + (A|B|F)^* + (C|B|E)^*.$$

Comparing the sum of equations 4.2, 4.3, 4.4 with equation 4.1 we conclude. \square

5. THE BIGRADED MODEL OF THE BARRATT ECCLES COMPLEX

In this section we start building the bigraded model of the Barratt-Eccles algebra $\mathcal{E}_2^*(4)$ and verify its non-formality. Let (TW, D) be the model of $\mathcal{E}_2^*(4)$ of Theorem 3.7, obtained by deforming the bigraded model $(TW, d) \simeq H^*(F_4)$, and equipped with a quasi-isomorphism $\phi : (TW, D) \rightarrow \mathcal{E}_2^*(4)$.

By Theorem 3.7 $d = D$ on W_0 and W_1 . We can characterize it explicitly.

Lemma 5.1. *On W_0 , $D = d = 0$. On W_1 , $D = d$ is determined by*

$$d((B_{ij}B_{kl})^*) = \begin{cases} (B_{ij}^*)^2 & \text{if } i = k \text{ and } j = l \\ B_{ij}^*B_{kl}^* + B_{kl}^*B_{ij}^* = [B_{ij}^*, B_{kl}^*] & \text{for } j < l \\ B_{ij}^*B_{kl}^* + B_{il}^*B_{ki}^* + B_{kl}^*B_{ki}^* & \text{for } j = l \text{ and } k < i \\ B_{ij}^*B_{kl}^* + B_{il}^*B_{ik}^* + B_{kl}^*B_{ik}^* & \text{for } j = l \text{ and } k > i \end{cases}$$

Proof. The product of two generators $B_{ij}B_{uv}$ of the Yang-Baxter algebra is a standard basis generator if and only if $j \leq v$. Instead, for $j > v$, the Yang Baxter relations yield

$$B_{ij}B_{uv} = \begin{cases} B_{uv}B_{ij} & \text{if } \{i, j\} \cap \{u, v\} = \emptyset \\ B_{uv}B_{ij} + B_{uj}B_{vj} + B_{vj}B_{uj} & \text{otherwise} \end{cases}$$

By dualizing the formula on basis generators we obtain the result. \square

We shall show that d and D do not agree on W_2 , and there is no “gauge equivalence” transformation fixing the problem.

Following the construction of the bigraded model in [7, 9] we can assume that ϕ sends each generator in $W_0 \cong H^1(F_4)$ to a cocycle representative.

Definition 5.2. Let $\phi : W_0 \rightarrow \mathcal{E}_2^*(4)$ be the linear map given by

$$\phi(B_{ij}^*) = \omega_{ij}.$$

We move to the next set of generators in filtration 1

Definition 5.3. Let $\phi : W_1 \rightarrow \mathcal{E}_2^1(4)$ be the linear map given by

$$\phi((B_{ij}B_{kl})^*) = \begin{cases} 0 & \text{if } (i, j) = (k, l) \\ \omega_{ij} \cup_1 \omega_{kl} & \text{if } j < l \\ \pi_{kil}^*(Ar) & \text{if } i > k \text{ and } j = l \\ \pi_{ikl}^*(Ar) + \omega_{il} \cup_1 \omega_{kl} & \text{if } i < k \text{ and } j = l. \end{cases}$$

We remark that the \cup_1 product of 1-cochains over \mathbb{Z}_2 has a simple form. Each k -cochain c over \mathbb{Z}_2 is determined by its support $Supp(c)$, the set of non-degenerate k -chains sent to 1 by c (rather than 0). Now for given 1-cochains c, c' the relation $Supp(c \cup_1 c') = Supp(c) \cap Supp(c')$ completely determines $c \cup_1 c'$. The construction of \cup_1 goes back to Steenrod, and is presented by McClure-Smith in [12].

The homomorphism $\pi_{kil} : W\Sigma_4 \rightarrow W\Sigma_3$ is induced by the simplicial version of the forgetful projection from 4 to 3 configuration points that we describe below (the projection to 2 points has been described in section 4).

We need a function $\pi_{kil} : \Sigma_4 \rightarrow \Sigma_3$ telling how a permutation twists the indices $\{k, i, l\}$. Consider the three symbols $j_1 = k, j_2 = i, j_3 = l$. A permutation $\sigma \in \Sigma_4$ gives a sequence $(\sigma(1), \sigma(2), \sigma(3), \sigma(4))$ containing a unique ordered subsequence $(j_{\tau(1)}, j_{\tau(2)}, j_{\tau(3)})$, where $\tau \in \Sigma_3$. We set $\pi_{kil}(\sigma) = \tau$. The construction induces a map on the W-construction still denoted $\pi_{kil} : W\Sigma_4 \rightarrow W\Sigma_3$, and then on normalized cochains

$$\pi_{kil}^* = N^*(\pi_{kil}) : \mathcal{E}_t^*(3) \rightarrow \mathcal{E}_t^*(4).$$

Lemma 5.4. *The homomorphism Φ of Definition 5.2 and 5.3 is compatible with the differentials, and thus defines a homomorphism of DGA $\phi : (T(W_{\leq 1}), d) \rightarrow \mathcal{E}_2^*(4)$.*

Proof. We have that $d((B_{ij}^2)^*) = (B_{ij}^*)^2$ by lemma 5.1

In the first case of 5.3

$$\phi(d((B_{ij}^2)^*)) = \phi((B_{ij}^*)^2) = \omega_{ij}^2 = 0 = d(\phi((B_{ij}^2)^*)).$$

Namely the top-dimensional 1-cocycle $(12|21)^* \in \mathcal{E}_2^1(2)$ squares to zero by dimensional reasons, and then so does its pullback $\omega_{ij} \in \mathcal{E}_2^1(3)$. In the second case of 5.3 by Lemma 5.1

$$d((B_{ij}B_{kl})^*) = [B_{ij}^*, B_{kl}^*] \text{ if } j < l, \text{ and}$$

$$\phi(d((B_{ij}B_{kl})^*)) = \phi([B_{ij}^*, B_{kl}^*]) = \omega_{ij} \cup \omega_{kl} + \omega_{kl} \cup \omega_{ij} = d(\omega_{ij} \cup_1 \omega_{kl})$$

by Steenrod's formula [12]. By definition

$$d(\omega_{ij} \cup_1 \omega_{kl}) = d(\phi((B_{ij}B_{kl})^*)).$$

In the third case of 5.3 $j = l$ and $i > k$ we have from Lemma 5.1 that

$$d((B_{ij}B_{kl})^*) = B_{ij}^*B_{kl}^* + B_{ij}^*B_{ki}^* + B_{kj}^*B_{ki}^*$$

and ϕ maps the right hand side to

$$\omega_{il}\omega_{kl} + \omega_{il}\omega_{ki} + \omega_{kl}\omega_{ki} = \pi_{kil}^*(d(Ar)) = d(\pi_{kil}^*(Ar)) = d(\phi((B_{ij}B_{kl})^*)).$$

In the last case of 5.3 for $i < k$ we have the same formula

$$d((B_{ij}B_{kl})^*) = B_{ij}^*B_{kl}^* + B_{ij}^*B_{ik}^* + B_{kj}^*B_{ik}^*$$

(we only swap i and k to have the term B_{ik} in standard form), so

$$\phi(d((B_{ij}B_{kl})^*)) = \omega_{il}\omega_{kl} + \omega_{il}\omega_{ik} + \omega_{kl}\omega_{ik} = \pi_{ikl}^*(d(Ar)) + [\omega_{il}, \omega_{kl}] =$$

$$= d(\pi_{ikl}^*(Ar) + \omega_{il} \cup_1 \omega_{kl}) = d(\phi((B_{ij}B_{kl})^*)).$$

Of course the product on cochains is not commutative and this matters in the ordering of the indices in the Arnold relation. \square

Extension to the filtration 2 stage

We compute the “error term” $D(w) - d(w) \in T(W_0)$, for $w \in W_2$, following the proof of Theorem 3.7.

Since $d(w) \in T(W_{\leq 1})$, we have already defined $\phi(d(w))$, that is a cocyle, since ϕ commutes with d so far. Its cohomology class $[\phi(d(w))] \in H^2(\mathcal{E}_2^*(4)) \cong H^2(F_4)$ might not be trivial, preventing us from extending ϕ to $T(W_{\leq 2}, d)$.

Definition 5.5. Let $\iota : H^2(F_4) \rightarrow T_2(W_0)$ be the embedding into the quadratic part sending each standard generator in the A_{ij} ’s to the analogous standard generator in the B_{ij} ’s. We set then $D(w) = d(w) - \iota[\phi(d(w))]$.

Proposition 5.6. *There is an extension of $\phi : (TW_{\leq 2}, D) \rightarrow \mathcal{E}_2^*(4)$ commuting with the differentials.*

Proof. We need to define $\phi(w)$ for $w \in W_2$, as w varies in the standard basis. In cohomology $[\phi(D(w))] = [\phi(d(w))] - [\phi(d(w))] = 0$, since $\phi([\iota(x)]) = x$. So there exists a cochain $\phi(w) \in \mathcal{E}_2^1(4)$ with $d\phi(w) = \phi(D(w))$. \square

The inductive construction continues in an infinite number of stages, building the full model (TW, D) , but here we are concerned only with filtration ≤ 2 .

Explicitly the “error term” $\phi(d(w))$ is the degree one map

$$W_2 \xrightarrow{\mu_{YB}^*} W_1 \otimes W_0 \oplus W_0 \otimes W_1 \xrightarrow{\phi_1 \otimes \phi_0 + \phi_0 \otimes \phi_1} \mathcal{E}^1 \otimes \mathcal{E}^1 \xrightarrow{\cup} \mathcal{E}^2.$$

and its cohomology class $\alpha(w) := [\phi(d(w))]$ can be computed explicitly.

For example

$$\alpha((B_{24}B_{14}B_{34})^*) = A_{13}A_{24}.$$

The full description of α is given later.

The computation is achieved by evaluating the cocycles $\phi(d(w))$ on standard chain representatives for basis elements of the homology group $H_2(F_4)$ that has dimension 11. Each representing cycle will be the sum of 8 basic generators.

We need to recall some facts on operads with multiplication in order to describe the cycles.

Definition 5.7. [14] A differential graded operad with multiplication over a fixed field R is a sequence of R -vector spaces $O(k)$ endowed with composition operations $\circ_i : O(k) \otimes O(l) \rightarrow O(k + l - 1)$ for $1 \leq i \leq k$ and multiplication operations $O(k) \times O(l) \rightarrow O(k + l)$ sending $(x, y) \mapsto x \cdot y$, satisfying appropriate axioms.

The Barratt Eccles operad $\mathcal{E}_2(k)^*$ forms a DGO with multiplication (over \mathbb{Z}_2 in this paper), and its homology $H_*(\mathcal{E}_2(k)^*) \xrightarrow{\theta} H_*(F_k)$ too [2].

We use lemma 6 in [13] that gives explicit homology representatives of the basis of $H_2(F_4)$ dual to the standard basis of $H^2(F_4)$.

Each homology generator is the image of the fundamental class of the torus via a map $(S^1)^2 \rightarrow F_4$.

Let us start from the homology generator $a \in H_1(F_2) \cong H_1(S^1) \cong \mathbb{Z}_2$ representing the orbit of a point labelled 2 rotating around a point labelled 1. The generator $(A_{12}A_{23})^*$ of $H_2(F_3)$ is the operadic composition $a \circ_2 a$. Pictorially the

map from the torus represents the configuration space of a planetary system with the sun labelled 1, the planet labelled 2, and the satellite labelled 3. The cycle $\gamma = (12|21) + (21|12) \in \mathcal{E}_2^1(2)$ represents the generating class $[\gamma] \in H_1(\mathcal{E}_2(2))$ corresponding to a under the isomorphism θ . Let us compute the class $[\gamma \circ_2 \gamma] = [\gamma] \circ_2 [\gamma] \in H_2(\mathcal{E}_2(3))$ corresponding to $a \circ_2 a$. In the Barratt-Eccles operad the composition is defined by appropriate substitution and relabelling [2]. For example

$$(12|21) \circ_2 (21|12) = (132|123|231) + (132|321|231).$$

This implies that the cycle $\gamma \circ_2 \gamma$ is a sum of 8 simplexes.

The class $(A_{12}A_{23})^* \in H_2(F_4)$ is the multiplication $(a \circ_2 a) \cdot u$ by the base point generator $u \in H_0(F_1)$. In the Barratt Eccles, on the chain level, the multiplication is obtained by sticking a 4 at the end of each permutation. The effect of this operation on $\gamma \circ_2 \gamma$ gives the first representative. The next homology generator $(A_{12}A_{13})^*$ is dual to a system where the sun is labelled 2, a planet 1, and a satellite 3. This homology class is the effect of the (12)-action on the previous class (exchange 2 and 1). This makes sense in any operad, and on the chain level in the Barratt Eccles operad as well, yielding the second representative. Proceeding in this way we obtain 8 basis elements out of 11, just by permuting labels. Namely each triple of indices out of 4 yields two generators. The three remaining classes are represented each by two independent planetary systems: the class $(A_{12}A_{34})^* = a \cdot a$ is represented by a map from the torus describing a configuration space where the star 1 has a planet 2, the star 3 has a planet 4, and the two systems are far apart. On the chain level it is represented by $\gamma \cdot \gamma$. The multiplication in the Barratt Eccles operad follows from the operadic structure, as $x \cdot y = (m \circ_2 y) \circ_1 x$, where $m = (12)$. For example

$$(12) \cdot (21) = (1234|2134|2143) + (1234|1243|2143),$$

and so in total $\gamma \cdot \gamma$ has 8 summands. The remaining 2 cycles representatives are the results of the action on $\gamma \cdot \gamma$ by (13) and (14) respectively.

Example 5.8. We show that

$$\alpha((B_{12}B_{23}B_{13})^*) = A_{12}A_{13} + A_{12}A_{23}.$$

The first part of the coproduct gives

$$\mu_{1,0}^*((B_{12}B_{23}B_{13})^*) = (B_{12}B_{23})^* \otimes B_{13}^* + (B_{23}B_{13} + B_{12}B_{13} + B_{12}B_{23})^* \otimes B_{12}^*.$$

Notice that

$$\Phi_1((B_{12}B_{23})^*) = \omega_{12} \cup_1 \omega_{23} = (123|321)^*$$

has trivial cup product with ω_{12} and ω_{13} by the complexity 2 hypothesis. Similarly

$$\Phi_1((B_{12}B_{13})^*) \cup \Phi_0(B_{12}^*) = (\omega_{12} \cup_1 \omega_{13}) \cup \omega_{12} = 0.$$

The only class contributing is then $(B_{23}B_{13})^* \otimes B_{12}^*$, mapping via $\Phi_1 \otimes \Phi_0$ to $\pi_{123}^*(Ar) \otimes \omega_{12}$. The cup product $\pi_{123}^*(Ar) \cup \omega_{12}$ pairs non-trivially with a 2-simplex such that the relative position of 1 and 2 is (12|12|21) by the complexity 2 requirement. The position of 1,2,3 in the first two permutations must match that of some generator of Ar . The possibilities are $(E|B)$ and $(E|A)$, but the second (312|123) does not appear in a cycle representative of a homology class, because 3 is not allowed to jump both 1 and 2 (intuitively because 3 is not the star of a planetary system). So the only possibility is (312|132|213) that appears in the expansion of the cycle representative of $(A_{12}A_{13})^*$.

The second part of the coproduct gives

$$\mu_{0,1}^*((B_{12}B_{23}B_{13})^*) = B_{12}^* \otimes (B_{23}B_{13})^* + B_{23}^* \otimes (B_{12}B_{13})^*.$$

The first summand goes to $\omega_{12} \cup \pi_{123}^*(Ar)$, but there are no generators in Ar where 1 and 2 have position (21|21), so there is no contribution. The second summand goes to $\omega_{23} \cup (\omega_{12} \cup_1 \omega_{13})$. This forces positions (123|132|321). In fact 2 and 3 must exchange first, and then 1 must jump both 2 3 by the \cup_1 product. This simplex appears in the cycle representative of $(A_{12}A_{23})^*$.

Lemma 5.9. *There is no isomorphism $\psi : (TW, d) \rightarrow (TW, D)$ such that $\psi - id$ lowers filtration.*

Proof. By assumption $\psi = id$ on W_0 , $\psi(W_1) \subseteq W_1 \oplus W_0$ and $\psi(W_2) \subseteq W_2 \oplus W_1 \oplus W_0$.

The value of ψ on W_1 is determined by a homomorphism $f : W_1 \rightarrow W_0 \cong H^1(F_4)$ such that $\psi|_{W_1} = id + f$. Since $d = D$ on W_1 , and $D(W_0) = d(W_0) = 0$, for any such choice ψ obviously commutes with the differential up to filtration level 1.

Now to make ψ commute with the differential in filtration 2 we need to construct $\psi_2(v) = v + w$, with $w \in W_1 \oplus W_0$. Now $D\psi_2(v) = Dv + dw$ must equate $\psi_1(dv)$. This is possible if and only if the cocycle $Dv - \psi_1(dv)$, for any $v \in W_2$, is a coboundary. We have that $Dv - \psi_1(dv) = \iota(\alpha(v)) + \bar{\beta}(f)(v)$ where

$\bar{\beta}(f)$ is the composition

$$W_2 \longrightarrow (W_1 \otimes W_0) \oplus (W_0 \otimes W_1) \xrightarrow{f \otimes 1 + 1 \otimes f} W_0 \otimes W_0 \xrightarrow{\mu} T_2(W_0).$$

Let $\beta(f)(v)$ be the cohomology class of $[\bar{\beta}(f)(v)]$, defining a homomorphism

$$\beta : Hom^0(W_1, H^1) \rightarrow Hom^1(W_2, H^2).$$

In cohomology we need $[Dv - \psi_1(dv)] = \alpha(v) + \beta(f)(v)$ to vanish for any $v \in W_2$, so we need to find f such that $\beta(f) = \alpha$.

Notice that $\dim(W_1) = 25$, $\dim(H^1) = 6$, $\dim(W_2) = 90$, $\dim(H^2) = 11$.

The computer verifies that $\beta : Hom^0(W_1, H^1) \rightarrow Hom^1(W_2, H^2)$, going from a 150-dimensional to a 990-dimensional vector space, has rank 137, but the span of its image and the obstruction cocycle α is a 138-dimensional subspace. This proves the lemma. \square

Proposition 5.10. *$F_4(\mathbb{R}^2)$ is not formal over \mathbb{Z}_2 .*

Proof. If F_4 were formal, by Proposition 3.8 there would be an isomorphism

$\Psi : (T(W), d) \cong (T(W), D)$, with $\Psi - id$ lowering filtration, but this contradicts lemma 5.9. \square

Here is an explicit description of the obstruction cocycle α . It sends generators on the left hand side to the respective elements on the right hand side, and is trivial on all other generators.

$$\begin{array}{ll}
(B_{12}B_{23}B_{13})^* & A_{12}A_{13} + A_{12}A_{23} \\
(B_{12}B_{13}B_{23})^* & A_{12}A_{13} + A_{12}A_{23} \\
(B_{12}B_{24}B_{14})^* & A_{12}A_{14} + A_{12}A_{24} \\
(B_{12}B_{14}B_{24})^* & A_{12}A_{14} + A_{12}A_{24} \\
(B_{23}B_{13}B_{13})^* & A_{12}A_{13} + A_{12}A_{23} \\
(B_{13}B_{23}B_{13})^* & A_{12}A_{23} \\
(B_{13}B_{13}B_{23})^* & A_{12}A_{13} \\
(B_{23}B_{13}B_{23})^* & A_{12}A_{13} \\
(B_{13}B_{23}B_{23})^* & A_{12}A_{13} \\
(B_{13}B_{34}B_{14})^* & A_{13}A_{14} + A_{13}A_{34} \\
(B_{23}B_{34}B_{24})^* & A_{23}A_{24} + A_{23}A_{34} \\
(B_{13}B_{14}B_{34})^* & A_{13}A_{14} + A_{13}A_{34} \\
(B_{23}B_{24}B_{34})^* & A_{23}A_{24} + A_{23}A_{34} \\
(B_{24}B_{14}B_{14})^* & A_{12}A_{14} + A_{12}A_{24} \\
(B_{34}B_{14}B_{14})^* & A_{13}A_{14} + A_{13}A_{34} \\
(B_{14}B_{24}B_{14})^* & A_{12}A_{24} \\
(B_{14}B_{34}B_{14})^* & A_{13}A_{34} \\
(B_{24}B_{34}B_{14})^* & A_{13}A_{24} \\
(B_{14}B_{14}B_{24})^* & A_{12}A_{14} \\
(B_{24}B_{14}B_{24})^* & A_{12}A_{14} \\
(B_{34}B_{14}B_{24})^* & A_{13}A_{24} \\
(B_{14}B_{24}B_{24})^* & A_{12}A_{14} \\
(B_{34}B_{24}B_{24})^* & A_{23}A_{24} + A_{23}A_{34} \\
(B_{14}B_{34}B_{24})^* & A_{13}A_{24} \\
(B_{24}B_{34}B_{24})^* & A_{23}A_{34} \\
(B_{14}B_{14}B_{34})^* & A_{13}A_{14} \\
(B_{24}B_{14}B_{34})^* & A_{13}A_{24} \\
(B_{34}B_{14}B_{34})^* & A_{13}A_{14} \\
(B_{24}B_{24}B_{34})^* & A_{23}A_{24} \\
(B_{34}B_{24}B_{34})^* & A_{23}A_{24} \\
(B_{14}B_{34}B_{34})^* & A_{13}A_{14} \\
(B_{24}B_{34}B_{34})^* & A_{23}A_{24}
\end{array}$$

Proposition 5.11. $F_k(\mathbb{R}^2)$ is not formal over \mathbb{Z}_2 for $k > 4$.

Proof. Suppose by contradiction that for $k > 4$ the configuration space F_k is formal over \mathbb{Z}_2 . Recall that $H^*(F_k)$ has as Koszul dual the algebra YB_k .

The filtered dual of the bar construction $B(YB_k)^*$ as seen earlier is the bigraded model of $H^*(F_k)$, and its homology is identified in a standard way with it. If F_k is \mathbb{Z}_2 -formal, then there is a quasi-isomorphism

$$B(YB_k)^* \xrightarrow{\simeq} C^*(F_k)$$

inducing the identity in cohomology [7].

Consider the projection $\pi : F_k \rightarrow F_4$ forgetting all points with labels larger than 4. It is easy to see that this map admits a section $s : F_4 \rightarrow F_k$.

Consider the DGA map $\bar{\pi} : YB_k \rightarrow YB_4$, defined purely algebraically, that forgets all generators with indexes larger than 4.

The homomorphism $\pi_{\#} : P\beta_k \rightarrow P\beta_4$ of fundamental groups induces via the descending central series a Lie algebra homomorphism $\pi_{\flat} : \mathcal{L}_k \rightarrow \mathcal{L}_4$, that after

applying the functor U gives back $\bar{\pi}$, modulo the identification $YB_k \cong U\mathcal{L}_k$ of Theorem 2.9.

Consider now the composite homomorphism of DGA's

$$B(YB_4)^* \xrightarrow{B(\bar{\pi})^*} B(YB_k)^* \xrightarrow{\simeq} C^*(F_k) \xrightarrow{s^*} C^*(F_4).$$

It induces identity on homology in degree 1, and henceforth in all degrees. This is a contradiction because F_4 is not formal over \mathbb{Z}_2 . \square

Propositions 5.10 and 5.11 together prove Theorem 1.3.

Open questions: is $F_4(\mathbb{R}^2)$ non-formal over \mathbb{Z}_p for some odd prime p ? Does this happen for infinitely many primes? How far do we have to go up in the bigraded model to find an obstruction? The same questions hold for $F_k(\mathbb{R}^2)$ and $k > 4$.

Notice that rational formality of a *simply connected* space implies \mathbb{Z}_p -formality for all primes p but a finite number (Theorem 3.1 in [7]). However this does not apply in our case, since $F_k(\mathbb{R}^2)$ is a $K(\pi, 1)$, so that rational formality of $F_k(\mathbb{R}^2)$ does not have implications for \mathbb{Z}_p -formality.

6. DEFORMATION THEORY AND HOCHSCHILD COHOMOLOGY

In this section we identify the obstruction class as an element in the Hochschild cohomology of the cohomology algebra of the configuration space.

Let us write $H = H^*(F_k)$ and $W = s(YB_k)^*$.

The graded vector space W concentrated in degree 1 splits as $W = \oplus_t W_t$, with W_t suspended dual of the space generated by words of length $t + 1$.

Definition 6.1. The degree 0 projection $\tau : W \rightarrow H$ sending $s(B_{ij})^* \mapsto A_{ij}$ and all other generators to zero is a *twisting cochain*.

Definition 6.2. The *convolution algebra* structure on $\text{Hom}(W, H)$ uses the coproduct structure on the domain and the product on the range to obtain an associative product of degree 1 [4]. For $f, g : W \rightarrow H$, we have the composition

$$f \star g : W \rightarrow W \otimes W \xrightarrow{f \otimes g} H \otimes H \rightarrow H.$$

Lemma 6.3. $\tau \star \tau = 0$.

The lemma follows from the coproduct structure described in lemma 5.1 and the relations holding in the cohomology algebra H .

There is a differential on the convolution algebra $\partial(f) = f \star \tau + \tau \star f$ that indeed squares to zero by lemma 6.3.

The complex $C = \text{Hom}(W, H)$ has a bigrading with $C^{p,q} = \text{Hom}(W_{p-1}, H^{q+1})$, that is inherited by its cohomology $H(C^{*,*}, \partial)$.

Proposition 6.4. [4] *There is a bigraded isomorphism*

$$H(C^{*,*}, \partial) \cong HH^{*,*}(H)$$

to the Hochschild cohomology of the cohomology algebra $H = H^(F_k)$.*

Hence the obstructions to formality live in the Hochschild cohomology groups $HH^{*,1}(H)$, see [15].

Proposition 6.5. *We constructed a non-trivial obstruction class*

$$0 \neq [\alpha] \in HH^{3,1}(H^*(F_4)) = (\mathbb{Z}_2)^{435}.$$

We verified using MATLAB that the homomorphism $\partial : C^{2,0} \rightarrow C^{3,1}$ has rank 137, and the homomorphism $\partial : C^{3,1} \rightarrow C^{4,2}$ has rank 418, so that the dimension of $HH^{3,1}$ is $990 - 137 - 418 = 435$.

7. APPENDIX: USER'S GUIDE TO THE ATTACHED MATLAB LIBRARY

Non-degenerate simplices of the Barratt-Eccles simplicial complex $W\Sigma_k$ are sequences of permutations, and are visualized as 2-dimensional arrays, or matrices. Each row represents a permutation. So a generator of $\mathcal{E}_i^j(k)^*$ is represented by a $(i+1) \times k$ matrix. Elements of the Barratt-Eccles complex are represented (non-uniquely) as 3 dimensional arrays, with a parameter in the third dimension for each generator. For example the basic 1-cocycle $\omega \in \mathcal{E}_2^1(2)$ is the dual of $[1\ 2; 2\ 1]$. The sum of these elements is performed by the function **ag** that cancels out redundant elements mod 2. The function **dbarr** computes the homology differential $\partial : \mathcal{E}_2^2(k)^* \rightarrow \mathcal{E}_2^1(k)^*$, together with basis of domain and codomain. These are given by 3-dim. arrays, the linear combination of all generators. Precisely

$$[M, B, A] = \text{dbarr}(k)$$

gives the basis A of $\mathcal{E}_2^1(k)^*$, the basis B of $\mathcal{E}_2^2(k)^*$, and the matrix M of the differential ∂ with respect to the said basis.

In order to pull back 1-cochains via the projection maps $\pi_V : W\Sigma_n \rightarrow W\Sigma_v$, for V a vector of v distinct elements of $\{1, \dots, n\}$, we use the routine **pullback**. If $x \in \mathcal{E}_2^1(v)$ then $\pi_V^*(x) \in \mathcal{E}_2^1(n)$ is obtained as **pullback**(n, V, x). The cup product $x \cup y \in \mathcal{E}_2^2(4)$ of two elements $x, y \in \mathcal{E}_2^1(4)$ is computed via **cup2**(x, y).

Instead the \cup_1 product $x \cup_1 y \in \mathcal{E}_2^1(4)$ of the 1-cochains x, y is computed by the routine **cupone**(x, y).

The homogeneous degree n summand $YB_k < n >$ of the Yang-Baxter algebra YB_k has a standard basis computed by **ybase**(k, n). A generator is a $2 \times n$ matrix so that $B_{i_1 j_1} \dots B_{i_n j_n}$ is represented by $[j_1 \dots j_n; i_1 \dots i_n]$ (mind the inversion), and the full basis is a 3-dim. array.

The function **yan** reduces in standard form a linear combination of products that are not necessarily standard, using the Yang-Baxter relations.

The function

$$[M, A, B, C] = \text{homalg}(i, j, k)$$

computes the multiplication in YB_k with respect to the standard basis: A, B, C are the basis in degree $i, j, i+j$ and M is a 3-dim. array describing the 3-tensor of the multiplication $YB_k < i > \otimes YB_k < j > \longrightarrow YB_k < i+j >$.

Similarly the function **genera**(k, n) provides the standard basis of the n -dimensional homology group of the planar configuration space on k points, and **arn** reduces sums of products to the standard form.

The more specialized routine $[E, V] = \text{massey2}(4)$ provides the obstruction cocycle in $Hom^1(W_2, \mathcal{E}_2^2(4))$. The result is a cell E with 90 entries, since $W_2 = (YB_4)_2^*$ is 90-dimensional. Each entry is a 2-cochain. Instead V is a cell of vectors

that are the coordinates of the elements of E with respect to the standard basis of $\mathcal{E}_2^2(4)$.

The function **barrcycle** provides cycle representatives for the standard basis of the homology group $H_2(F_4(\mathbb{R}^2))$ of rank 11, and each cycle is the sum of 8 simplexes, so it produces a cell with 11 entries, and each is a $3 \times 4 \times 8$ 3-dimensional array.

Then the function $A = \mathbf{terminenoto}(E)$ evaluates the previous cocycles E , output of **massey2**, against the cycles from **barrcycle**, producing a 90×11 matrix of homology classes representing the obstruction $\alpha \in \text{Hom}^1(W_2, H^2)$.

Finally the function $[Z, M, N, C] = \mathbf{sistemone}(A)$ inputs the output of **terminenoto**. It resizes it as a column C with 990 elements. Then M is a 990×150 matrix representing the homomorphism $\beta : \text{Hom}^0(W_1, H^1) \rightarrow \text{Hom}^1(W_2, H^2)$. This homomorphism splits in two parts using the two summands of the comultiplication $W_2 \rightarrow (W_1 \otimes W_0) \oplus (W_0 \otimes W_1)$. They are juxtaposed to form a 990×300 matrix N . The 150-components vector Z is one solution of the equation $MZ = C$ if it exists. However it is not found because it does not exist.

The subfunction used here is $Z = \mathbf{risolvi}(M, C)$ that solves linear systems over \mathbb{Z}_2 . A related function is $r = \mathbf{rango}(A)$ that computes the rank over \mathbb{Z}_2 of the matrix A .

Other subfunctions that are included are: **recog**, **be**, **subset**, **mono**, **inverse**.

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